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Thermodynamics at High Energies

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A new thermodynamics, applicable to cosmic ray showers and high-energy physics, is developed. Although all density expressions are unaltered, their global forms are modified due to the new dependence between the volume and the temperature. This occurs in bound systems where the number of particles, instead of being an increasing function of the temperature, is a decreasing one. That certain global expressions for the entropy turn out to be convex functions of the energy necessitates their reinterpretation as the reduction in entropy caused by the volume–temperature constraint. The continuous distribution for the production of hadrons with energies greater than a given amount is shown to correspond to the fact that discrete particle fluctuations follow Poisson's law.

1. EXTREME-VALUE DISTRIBUTION OF COSMIC RAY SHOWERS

Prior to the advent of particle accelerators, cosmic rays were the primary source of data for high-energy particle physics. A hallmark of primary cosmic radiation is that instead of monoenergetic particles, it has an energy spectrum of primary protons and other nuclei that may be represented as (Rossi, 1955)

$$\nu(E) = \frac{A}{E^{\sigma}}$$

where $\nu(E)$ is the number of secondary particles having an energy greater than the low-energy cutoff of the spectrum. This form of the spectrum with $\sigma = 5/2$ has been used for all energies >2 GeV (Rossi, 1955). The constant A is related to the energy of the primary particle, E' by $A = E'^{\sigma}$. If ν_0 is the maximum number of particles that can be produced in a shower, then $\nu_0 = (E'/E_0)^{\sigma}$. The minimum energy that a secondary particle can possess,

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E_0, is the low-energy, cutoff of the spectrum, which takes place at about 58° geometric latitude, corresponding to a proton energy of 5.6×10^8 eV (Ginzburg, 1958).

Introducing the maximum number of particles into the spectral distribution yields

$$\frac{\nu}{\nu_0} = \left(\frac{E_0}{E}\right)^{\sigma} \tag{1}$$

which can be interpreted as the tail of a probability distribution. In the course of time, the power law (1) can be modified by ionization, statistical acceleration, dissipative losses, and bremsstrahlung. For cosmic protons the nuclear magnetic bremsstrahlung and ionization losses will be small in the relativistic range, so that the spectral energy density retains its power-law form (Ginzburg, 1958).

However, the initial tail distribution (1) can change in other, nondeterministic ways, simply with the increase in the number of particles in the cosmic ray shower. In fact, the tail distribution (1) will be "attracted" asymptotically to a limit distribution other than that dictated by the central limit theorem. The limit distribution will turn out to be the order-statistic generalization of an extreme-value distribution (Lavenda, 1996), since the initial distribution (1) has the form of a Pareto–Lévy distribution familiar from economics.

If hadron statistical physics followed the law of large numbers and the central limit theorem, then the probability that the energy would lie between E and $E + dE$ would be given by (Lavenda, 1991, §4.2)

$$h(E; \nu)\, dE = \frac{\beta(\beta E)^{\nu-1}}{\Gamma(\nu)} e^{-\beta E}\, dE \tag{2}$$

where β is the inverse temperature in energy units where Boltzmann's constant equals unity. The first moment of the gamma density (2),

$$\beta \overline{E} = \nu \tag{3}$$

where $\overline{E}$ is the average energy and ν is half the number of degrees of freedom, ensures the equipartition of energy. However, the law of equipartition of energy requires particle conservation, which is certainly not fulfilled at high temperatures exceeding those of particle thresholds. Moreover, equipartition states that the average kinetic energy is proportional to the energy itself, leaving no contribution to particle creation.

Thus, we need a generalization of the law of equipartition of energy (3) that would allow for variable dependences of the temperature of the energy. This leads us to consider

$$\beta \overline{E} = \nu(\overline{E}) \tag{4}$$

where the number of particles of energy $>\overline{E}$ has the form (1).

The power law for the number of particles having energies $>E$ also has the correct form for the multiplicity of secondary hadrons formed from hadron–nucleon collisions, when it is taken to be a function of the energy of the incident hadron E'. It has been experimentally established that the multiplicity of secondary hadrons produced in the collision of hadrons and other particles with nucleons increases with increasing energy of the primary hadron and has the form $\nu = (E'/E)^{\sigma}$, for energies in the center-of-mass frame >10 GeV, with an exponent $\sigma = 0.4$ (Kalinoskii *et al.*, 1989).

Following Bhabha and Heitler (1937), we measure energy by means of the dimensionless quantity

$$y = \ln \frac{E}{E_0} \tag{5}$$

Let us assume for the moment that the relation between E and y is of the same form as (4), namely

$$\beta E = \nu_0 e^{-\sigma y} \tag{6}$$

which will have maximum probability of occurrence for the value $\bar{E}$. Then, if (6) is employed as a transformation of the independent variable, we obtain a new distribution

$$h(y; n)\, dy = \sigma \frac{\nu_0^{\nu}}{\Gamma(\nu)} \exp\{-\nu\sigma y - \nu_0 e^{-\sigma y}\}\, dy \tag{7}$$

which is a double-exponential distribution for the νth largest value (Cramér, 1946). Provided ν is sufficiently large that Stirling's approximation is applicable, (7) will have a sharp maximum at

$$\bar{y} = \frac{1}{\sigma} \ln \frac{\nu_0}{\nu} \tag{8}$$

This simply says that the thermal equation of state (4) maximizes the probability.

The basis of the transformation from the gamma distribution (2) to the double-exponential distribution (7) rests on the theory of order statistics, and in particular on a method introduced by Cramér (1946). Suppose we are sampling values of the energy from a population with a continuous distribution $F(E)$ whose density is $f(E)$. The probability that, in a population of ν_0 particles, $\nu_0 - \nu$ have energies $<E$ and $\nu - 1$ particles have energies $> E + dE$, with the remaining particle energies falling in the interval from E to $E + dE$, is given by the beta distribution

$$g(E; \nu)\, dE = \frac{1}{B(\nu, \nu_0 - \nu + 1)} F^{\nu_0 - \nu}(E)[1 - F(E)]^{\nu - 1} f(E)\, dE \tag{9}$$

where B is the beta function. If the tail of the distribution falls off according to the power law

$$1 - F(E) = \left(\frac{E_0}{E}\right)^{\sigma} \tag{10}$$

then in the asymptotic limit, as the population grows without limit, the distribution will be attracted to the strictly stable law[2]

$$h(E;\, \nu)\, dE = \frac{\sigma \nu_0^{\nu}}{\Gamma(\nu)} \left(\frac{E_0}{E}\right)^{\sigma\nu} e^{-\nu_0 (E_0/E)^{\sigma}} \frac{dE}{E} \tag{11}$$

The derivation of the asymptotic distribution identifies particles with what we have previously referred to as half degrees of freedom. In high-energy physics, fireballs and low-lying resonances are essentially the same thing: excited hadrons (Hagedorn, 1965). The fact that the excited hadrons ν vary with their energy as $\nu \sim E^{-\sigma}$ could possibly be explained by Landau's observation (Belenkij and Landau, 1956) that at the moment of particle collision when the fireball is formed, the particle densities are so high and the interactions are so strong that the concept of an entity referred to as a "particle" loses meaning. It is only when the system expands, thereby decreasing both the density and temperature, that particles, or excitations, become discrete entities.

That ν also varies with the temperature as $\nu \sim T^{-\sigma/(1+\sigma)}$ is an additional hypothesis: since the nuclear interaction is very strong and the volume into which the energy is released is so small, thermal equilibrium should be achieved. This, in effect, is what allows us to determine the energy distribution by statistical laws (Fermi, 1950, 1951). The crucial point is that it is not the global volume housing the entire system, or what would be analogous to the *Hohlraumstrahlung* in blackbody radiation, that enters into the relevant thermodynamic expressions, but, rather, the volume, which is determined by the range of the strong interactions among the particles.

The probability distribution (11) is comparable with Fermi's (1950, 1951) expression for the probability of the multiple production of ν pions. Fermi started with the product of the independent statistical weighting functions of phase space and proceeded to the asymptotic limit. Fermi predicted a simple exponential decay in probability with increasing energy, while (11) has an exponential factor which increases with the energy. It is apparent from its derivation that Fermi's expression is not a probability, since it is not normalizable. If, like Fermi, we assume the particles to be statistically inde-

[2] Since the distribution is strictly asymptotic in the limit $\nu_0 \to \infty$, we would have to introduce the scaling coefficients s_{ν_0} in $F(y)$, *i.e.*, $F(s_{\nu 0}E)$, where $s_{\nu 0} = E_0 \nu_0^{1/\sigma}$. The scaling constants $s_{\nu_0} \to \infty$ the limit.

pendent and all with a common tail distribution given by (10), then order statistics asserts that the asymptotic probability of finding ν pions with energies $>E$ will be given by (11).

2. THERMODYNAMICS OF BOUND SYSTEMS

According to the generalization (4) of the equipartition law and the Pareto–Lévy tail, (1), the temperature increases faster than the average energy, $Y \sim \overline{E}^{(1+\sigma)}$, for $\sigma > 0$. This behavior is certainly not common in conventional thermodynamics. For an ideal material gas, the temperature increases with the energy itself, since the particle number is constant. For a photon gas, the temperature increases only as $T \sim E^{1/4}$, where the additional energy is available for the creation of photons. The fact that the temperature grows faster than the energy itself is a hallmark of *bound* thermodynamic systems. One should also realize that there is an inherent connection between binding and scattering; a bound state can be thought of as a continuous sequence of scatterings that keeps the particle from escaping to infinity (Gottfried and Weisskopf, 1982). Consequently, the scattering processes should manifest the same temperature dependence on the energy that bound systems do.

That the thermal equation of state (4) actually applies to a bound system can be shown by an application of the virial theorem for nonrelativistic systems (Jeans, 1929). For bound systems the total energy,

$$\mathcal{E} = \mathcal{T} + \Omega + \overline{E} \tag{12}$$

is negative, where $\mathcal{T}$ is the average kinetic energy, and $\Omega < 0$ can stand for any attractive potential such as the gravitational potential. Now, from the stationary virial, it follows that

$$2\mathcal{T} + \Omega = 0$$

so that the total energy reduces to

$$\mathcal{E} = \overline{E} - \mathcal{T}$$

Since $\mathcal{T} \sim T$, it follows that

$$\overline{E} < T \tag{13}$$

is the condition for a bound system, *i.e.*, $\mathcal{E} < 0$. Since relativistic effects should not affect the nature of bound systems by converting them into unbound ones, we argue that the same condition for bound systems also holds in the extreme ultrarelativistic limit even though it cannot be derived from the relativistic virial. In the extreme ultrarelativistic limit of a photon "star," the condition for hydrostatic equilibrium becomes independent of the radius, indicating a lack of stability.

Also implied in the thermal equation of state (4) is that what was formally identified as half the number of degrees of freedom has now acquired a temperature dependence. And this temperature dependence implies that the number of degrees of freedom decreases with increasing temperature. Counterintuitive results such as this one are typical of high-temperature, high-density regimes (Sertorio, 1979).

Application of the second law to the average equation of state will yield an expression for the entropy. For, according to the second law, the derivative of the entropy with respect to the energy is the inverse temperature:

$$\frac{dS}{d\overline{E}} = \frac{1}{T} = \frac{1}{\overline{E}}\left(\frac{E_0}{\overline{E}}\right)^{\sigma} \nu_0 \tag{14}$$

Integrating (14), we obtain a negative quantity which can hardly be identified with an entropy. The limits of traditional thermodynamics have been exceeded, and what we are dealing with in bound systems is not an entropy at all, but rather, the *reduction* in entropy (Lavenda, 1995)

$$\Delta S = -\frac{1}{\sigma}\left(\frac{E_0}{\overline{E}}\right)^{\sigma} \nu_0 \tag{15}$$

due to a constraint that has been placed upon the system.

In a general thermodynamic system in thermodynamic equilibrium the three state variables P, V, and T are related through a mechanical equation of state, so that only two or the three variables are independent. If an *additional* condition is imposed between two of the independent state variables, in this case between volume and temperature, then only one variable can be varied independently. The reduction in the number of independent thermodynamic variables is akin to a polytrope (Cox, 1968). That $E_0 < \overline{E}$ is indicative of particle interactions which increase the entropy reduction, which would otherwise be proportional to the negative of the number of particles, just as the entropy is proportional to the number of particles of an ideal degenerate gas. Conventional thermodynamics therefore has at least two limits: small systems in which Stirling's approximation fails (Lavenda, 1991), and bound systems (Lavenda, 1995).

In either bound or unbound systems the number of particles is proportional to the volume of phase space occupied by the system,

$$\nu \sim V/\lambda_T^3 \sim VT^{\eta} \tag{16}$$

provided there is no particle conservation, where λ_T is the thermal wavelength and η is the adiabatic exponent, or the number of half degrees of freedom. For a nonrelativistic monatomic gas $\eta = 3/2$, while for an ultrarelativistic gas $\eta = 3$. It will turn out that V is not the geometric volume enclosing the system, but, rather, the volume that is determined by range of the interactions.

In contrast to an unbound degenerate ideal gas, *at constant volume*, where

$$\nu \sim E^{\eta/(1+\eta)} \sim T^{\eta} \tag{17}$$

in a bound system, corresponding to a strictly stable law, we have

$$\nu \sim E^{-\sigma} \sim T^{-\sigma/(1+\sigma)} \doteq T^{-\alpha} \tag{18}$$

which defines the canonical strictly stable law exponent α, assuming values on the open interval (0, 1). A comparison of (17) with (18) would lead to the absurd conclusion that $\sigma = -\eta/(1 + \eta)$ or $\eta = -\sigma/1 + \sigma)$. This is to say that for $\sigma > 0$, the number of half degrees of freedom must be negative. The problem is that we have not allowed for variations in the volume with temperature in the case of bound systems.

The string of relations in (16) can be extended to

$$\nu \sim V/\lambda_T^3 \sim VT^{\eta} \sim E^{\mp\sigma} \sim T^{-\alpha} \tag{19}$$

in bound systems where the last relation in (19) follows from the application of the second law. The $\mp$ sign in the exponent of the energy applies to strictly stable and quasistable laws, respectively (Lavenda, 1996). The two are distinguished by the interval over which the canonical exponent α varies. As we have mentioned for strictly stable laws, the exponent α varies over the open interval (0, 1), whereas for quasistable laws, its range is (1, 2). The strictly stable law governs maximum values, while the quasistable law manages minimum values. Here, we will be concerned only with strictly stable laws for positive, maximum values.

From the chain of relations (19) it follows that

$$VT^{(\alpha+\eta)} = \text{const} \tag{20}$$

The adiabatic case $\alpha = 0$ separates bound systems, where $\alpha > 0$, from unbound systems, where $\alpha < 0$. Indeed, the condition for a bound system, (13), written as $\varepsilon V < T$, where the energy density $\varepsilon \sim T^{\eta+1}$, simply translates into the fact that $T^{1-\alpha} < T$ for $\alpha > 0$. Alternatively, if $V = \text{const}$, (19) implies that $\eta = -\alpha$, which gives $S \sim E^{\eta/(1+\eta)}$. This is identical to the expression for the entropy density [cf. (21) below], and so implies that $V = \text{const}$. Consequently, as α goes from the negative value $-\eta$ to positive values, either on the open interval (0, 1) or (1, 2),[3] the entropy is converted into an entropy reduction with the adiabatic case separating the two domains.

It must be emphasized that *density relations cannot discriminate between bound and unbound systems; only their global relations can because of the*

[3]The case $\alpha = 1$ must be handled separately and corresponds to the Cauchy distribution.

appearance of the volume in these expressions. For example, the entropy density is

$$s \sim \varepsilon^{\eta/(1+\eta)} \tag{21}$$

From (19) it follows that if the volume is constant, the particle number will *increase* with temperature as T^{η}. In contrast, for a bound system it will *decrease* with temperature as $T^{-\alpha}$. In either case the number density is left invariant, varying with temperature as $\nu/V \sim T^{\eta}$. This is probably why bound systems have gone undiscovered until now: their density relations are identical to those of unbound systems. However, the differences in the global relations imply that there are fundamental differences in the underlying statistics. In other words, for bound systems, the volume is elevated to a dependent variable, like that of the particle number, and not a mere parameter as in Bose–Einstein and Fermi–Dirac statistics. Hence, we anticipate that the configuration-space volume element will acquire a more prominent role than it has hitherto been given. In small volumes at high energies it is the momentum-space volume element that should thermalize prior to the configuration-space volume element, which should undergo expansion as the temperature drops (Belenkij and Landau, 1965). The momentum-space volume element will have then attained its minimum value, which is the inverse of the 'thermal' volume per particle, whose radius is the thermal wavelength.

For unbound systems with constant volume we can immediately proceed to replace the density relation (21) by its global one, $S \sim E^{\eta/(1+\eta)}$. However, for bound systems we must take into consideration the variation of the volume with temperature, (20). Consequently, the density relation (21) is equivalent to the global relation

$$\begin{aligned} S &\sim V^{1/(1+\eta)}E^{\eta/(1+\eta)} \\ &\sim E^{(\eta-\sigma)/(1+\eta)}T^{-\eta/(1+\eta)} \\ &\sim E^{-\sigma/(1+\eta)+\eta(1-\sigma/\alpha)/(1+\eta)} = E^{-\sigma} \end{aligned} \tag{22}$$

where we have used $\sigma = \alpha/(1-\alpha)$. Because the entropy density is a *concave* function of the energy density, so, too, should the entropy be a concave function of the energy. However, (22) is clearly a *convex* function. Consequently, we are led to consider the negative of that expression as the entropy reduction due to the constraint (20). The first moment will be identical since $dS/dE = ds/d\varepsilon$, while the second moment is

$$\overline{(\Delta E)^2}/V = \overline{(\Delta\varepsilon)^2}$$

because $d^2S/dE^2 = V^{-1}\, d^2s/d\varepsilon^2$.

3. PROPERTIES OF THE ASYMPTOTIC DISTRIBUTION

It will prove convenient to write the generating function of the double-exponential distribution (7) as

$$\mathfrak{X}^{\sigma}(\nu) = \left(\frac{\nu}{\nu_0}\right)^{\nu} e^{-\nu} = \int_0^{\infty} e^{-\nu\sigma y - \nu_0 e^{-\sigma y}}\, d(\sigma y) \tag{23}$$

rather than $\mathfrak{X}$ itself, because the independent variable y always appears in the company of σ, so that if we were to retain the usual definition of the generating function, then we would be forced to identify the independent variable as σy rather than y itself. The fact that we are free to do so reveals an important property of the distribution (7); namely, the probability law is *infinitely divisible* (Feller, 1968). Infinitely divisible distributions imply stochastic independence: the generating function factorizes on nonoverlapping intervals on the space on which it is defined.

Due to the power law structure inherent in (7), the generating function can be considered as the Laplace transform

$$\mathscr{L}(\nu) = \int_0^{\infty} e^{-\nu x}\Omega(x)\, dx$$

of the structure function

$$\Omega(y) = \sigma e^{-\nu_0 e^{-\sigma y}} \tag{24}$$

or as the Mellin transform

$$\mathscr{M}(\eta) = \int_0^{1} x^{\zeta-1}\Omega(x)\, dx$$

of the structure function

$$\Omega\left(\frac{E_0}{E}\right) = \sigma e^{-\nu_0 (E_0/E)^{\sigma}} \tag{25}$$

where the Mellin transform variable is $\zeta = \sigma\nu$. The logarithm of either structure function is proportional to the entropy reduction (15).

The logarithm of the generating function (23),

$$\ln \mathfrak{X}(\nu) = -\frac{\nu}{\sigma}\left[1 - \ln\left(\frac{\nu}{\nu_0}\right)\right] \tag{26}$$

is a completely monotone function, since its derivatives alternate in sign. For example, the first two derivatives are

$$\frac{d}{d\nu}\ln \mathfrak{X} = \frac{1}{\sigma}\ln\left(\frac{\nu}{\nu_0}\right) = -\bar{y} \tag{27}$$

and

$$\frac{d^2}{d\nu^2}\ln \mathfrak{X} = \frac{1}{\sigma\nu} = -\frac{d\bar{y}}{d\nu} > 0 \tag{28}$$

which are the first two central moments of the distribution. Expression (28) not only shows that the logarithm of the generating function is convex in the number of half degrees of freedom, but, in addition, that the average energy is a *decreasing* function of the degrees of freedom. This is yet another counterintuitive prediction for bound systems.

The structure of exponential distributions and the Legendre transform lead to the conclusion that there must exist a dual function to the logarithm of the generating function. The dual, which must turn out to be concave in the dimensionless energy variable $\bar{y}$, is defined by the Legendre transform:

$$\ln \mathfrak{X} - \frac{d}{d\nu}\ln \mathfrak{X} = -\frac{\nu_0}{\sigma}e^{-\sigma\bar{y}} \tag{29}$$

This function will be appreciated as the entropy reduction (15) when the transform (5) is made.

In Hagedorn's theory of high energies (Sertorio, 1979), the structure function increases as some exponential function of the energy with an exponent less than unity. This implies that for the convergence of the Laplace integral there must be a maximum temperature. The reason for introducing a maximum temperature is that every time a characteristic temperature range is surpassed, what was an elementary building block begins to show compositeness. Introducing a high-temperature cutoff simply means that there is a limit to the elementary constituents of matter (Sertorio, 1979). In contrast to Hagedorn's theory, where the number of particles increases with the energy as the temperature tends to a constant, we have assumed, following the experimental evidence of cosmic ray showers, that the number of particles surpassing a given energy is a decreasing function of that energy.

Hagerdorn's entropy *density*, $s \sim \varepsilon^{3/(3+\kappa)}$, is a concave function of the energy density ε over the half open interval $0 < \kappa < \infty$. The physical significance attributed to κ is that it is an ordering parameter for degeneracies (Sertorio, 1979), the case $\kappa = 1$ corresponds to a photon gas. The connection between the different values of κ and the different statistical counting of states in phase space is inaccurate because $s = s(\varepsilon)$ is an *integrated relation* for which there is no memory of the statistical counting procedure. We will now show that there is only a single value of κ which is acceptable, namely, $\kappa = 1$.

Consider the following string of relations:

$$S \sim VT^{\eta} \sim E^{\eta/(\eta+\kappa)} = (\varepsilon V)^{\eta/(\eta+\kappa)}$$

From this it follows that

$$T^{(\eta+\kappa)/\kappa} \sim \varepsilon^{1/\kappa} V^{-1/\eta} \sim \varepsilon$$

The last relation follows from the second law applied to the entropy density (21), namely $ds/d\varepsilon \sim \varepsilon^{-\kappa/(\eta+\kappa)} \sim T^{-1}$. This condition leads to the conclusion that $V \sim \varepsilon^{-\eta/\kappa(1-\kappa)}$. But the energy density cannot be a function of the volume for otherwise $E \doteq \varepsilon V \sim \varepsilon^{[\kappa-\eta(1-\kappa)]/\kappa}$, and this requires us to set $\kappa = 1$, implying that $V = \text{const}$.

4. FLUCTUATIONS IN PARTICLE NUMBER

Thus far we have considered the distribution for y with ν as a parameter and found the asymptotic distribution in energy to be driven by the double exponential (7). To add further support in favor of (11) as the asymptotic distribution for the probability that the ν particles will have energies $>E$, we transform it into a well-known probability law giving fluctuations in the number of particles ν. This follows from the rather remarkable property that when we invert the roles of y and ν, treating the latter as the variate and the former as the parameter, which we evaluate at maximum likelihood (27), we obtain a well-known discrete distribution for the degrees of freedom. Setting y at its maximum likelihood value has the effect of transforming the distribution into an *error law* (Lavenda, 1991, §4.11). Denoting by $\bar{\nu}$ that value in (27) and evaluating the double-exponential distribution (7) at $y = \bar{y}$ results in the well-known Poisson distribution

$$p(\nu;\bar{\nu}) = \frac{\bar{\nu}^{\nu}}{\nu!}e^{-\bar{\nu}} \tag{30}$$

The Poisson distribution of finding ν particles is assumed to apply for showers if the presence of one particle at any given depth is independent of the processes which precede it. It is generally believed that the Poisson law (30) greatly underestimates the fluctuations in the number of particles at any given depth because of the lack of statistical independence (Rossi, 1952). The presence of the Poisson law (30) could have been intuitively sensed from the fact that the negative of the logarithm of the generating function in (23) has the form of the function which determines the error law (Lavenda, 1991, p. 84). The error law for deviations from the most probable value $\bar{\nu}$ is an expression of the convexity of the logarithm of the generating function (23).

Finally, it should be emphasized that the assumption that particle fluctuations follow a Poissonian law is commonly considered to be an *additional* assumption based on the supposedly statistical independence of the emission processes (Rossi, 1952). Here, we have shown it to be a consequence of the fact that the probability of finding ν particles with energies $> E$ is given by an asymptotic distribution of order statistics for the largest values. It is a matter for experiment to discriminate between this distribution and the one derived by Fermi based on an asymptotic form of products of phase-space volume elements (Fast and Hagedorn, 1963; Fast *et al.*, 1963; Hagedorn, 1965; Auberson and Escoubes, 1965).

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